

# AN ALTERNATIVE PROOF OF HILL'S CRITERION OF FREEDOM FOR ABELIAN GROUPS

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**ABSTRACT.** In this note, we provide a different proof of Hill's criterion of freeness for abelian groups. Our proof hinges on the construction of suitable  $G(\aleph_0)$ -families of subgroups of the links in Hill's theorem and, ultimately, on the construction of such a family of pure subgroups of the group itself.

**RESUMEN.** En este trabajo, se proporciona una nueva demostración del criterio de Hill para grupos abelianos libres. La demostración se basa en la construcción de una  $G(\aleph_0)$ -familia de subgrupos en los eslabones del teorema de Hill y, prioritariamente, en la construcción de una familia tal de subgrupos puros.

## 1. INTRODUCTION

In 1934, Lev Pontryagin proved that a countable, torsion-free abelian group is free if and only if every finite rank, pure subgroup is free [3]. Equivalently, every properly ascending chain of subgroups of the same finite rank is finite. From the proof of this criterion, it follows that a torsion-free abelian group  $G$  is free if there exists an ascending chain

$$(1) \quad 0 = G_0 < G_1 < \cdots < G_n < \cdots \quad (n < \omega),$$

consisting of pure subgroups of  $G$  whose union is equal to  $G$ , such that every  $G_n$  is free and countable. Here, a subgroup  $H$  of the abelian group  $G$  is *pure* if solubility in  $G$  of every equation of the form  $nx = h \in H$ , with  $n \in \mathbb{Z}$ , implies its solubility in  $H$ . Also, we say that  $G$  is *torsion-free* if  $n = 0$  or  $g = 0$ , whenever  $n \in \mathbb{Z}$  and  $g \in G$  satisfy  $ng = 0$ .

Later, in 1970, Hill established that, in order for an abelian group  $G$  to be free, it is sufficient to prove that it is the union of a countable ascending chain (1) consisting of free, pure subgroups [1]. In other words, he proved the following theorem, establishing thus that the countability condition on the cardinality of the links of the chain was superfluous.

**Theorem 1.1** (Hill's criterion of freeness). *A torsion-free abelian group  $G$  is free if there exists a countable ascending chain*

$$(2) \quad 0 = G_0 < G_1 < \cdots < G_n < \cdots \quad (n < \omega)$$

*of subgroups of  $G$ , such that:*

- (a) *every  $G_n$  is free,*
- (b) *every  $G_n$  is a pure subgroup of  $G$ , and*

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$$(c) \ G = \bigcup_{n < \omega} G_n.$$

In this note, we give a proof of Hill's criterion different from the one provided in [1]. Our proof hinges on the construction of suitable classes of subgroups of the groups  $G_n$  and, ultimately, on the construction of such a family consisting of pure subgroups of  $G$ . Section 3 of this work contains the proof of Theorem 1.1, while Section 2 presents some preliminary results.

## 2. PREPARATORY LEMMAS

The following is a general result which will be used in the proof of Theorem 1.1. We refer to [2] for definitions of the set-theoretical concepts.

**Lemma 2.1.** *An abelian group  $G$  is free if there exists a continuous, well-ordered, ascending chain*

$$(3) \quad 0 = A_0 < A_1 < \cdots < A_\gamma < A_{\gamma+1} < \cdots \quad (\gamma < \tau)$$

*of subgroups of  $G$ , such that:*

- (a) *every factor group  $A_{\gamma+1}/A_\gamma$  is free, and*
- (b)  $G = \bigcup_{\gamma < \tau} A_\gamma$ .

*Proof.* The conclusion follows from the fact that  $G$  is isomorphic to the direct sum of the factor groups  $A_{\gamma+1}/A_\gamma$ , for  $\gamma < \tau$ .  $\square$

Recall that a  $G(\aleph_0)$ -family of an abelian group  $G$  is a collection  $\mathcal{B}$  of subgroups of  $G$ , which satisfies the following properties:

- (i)  $0$  and  $G$  belong to  $\mathcal{B}$ ,
- (ii)  $\mathcal{B}$  is closed under unions of ascending chains, and
- (iii) for every  $A_0 \in \mathcal{B}$  and every countable set  $H \subseteq G$ , there exists  $A \in \mathcal{B}$  which contains both  $A_0$  and  $H$ , such that  $A/A_0$  is countable.

Clearly, every abelian group has a  $G(\aleph_0)$ -family, namely, the collection of all its subgroups.

For the rest of this section, we will assume the hypotheses of Theorem 1.1. Under these circumstances, we fix a basis  $X_n$  of  $G_n$  for every  $n < \omega$ , and let  $\mathcal{B}_n$  be the family of all subgroups of  $G_n$  generated by subsets of  $X_n$ . Clearly, every member of  $G_n$  is a direct summand of  $G_n$  and, thus, a pure subgroup of  $G$ .

**Lemma 2.2.** *The collection  $\mathcal{B}'_n = \{A \in \mathcal{B}_n : A + G_i \text{ is pure in } G, \text{ for every } i < \omega\}$  is a  $G(\aleph_0)$ -family of pure subgroups of  $G_n$ , for every  $n < \omega$ .*

*Proof.* All we need to check is that the countability condition is satisfied, since the other conditions of a  $G(\aleph_0)$ -family are obvious. So, let  $A_0 \in \mathcal{B}'_n$ , and let  $H_0$  be a countable subset of  $G_n$ . Moreover, let  $m < \omega$ , and assume that we have already constructed a chain

$$(4) \quad A_0 < A_1 < \cdots < A_m$$

of groups in  $\mathcal{B}_n$ , such that:

1.  $H_0$  is contained in  $A_1$ ,
2. for every  $j < m$ , the group  $A_{j+1}/A_j$  is countable, and
3. for every  $j < m$  and every  $i < \omega$ ,  $(A_{j+1} + G_i)/(A_0 + G_i)$  contains the purification of  $(A_j + G_i)/(A_0 + G_i)$  in  $G/(A_0 + G_i)$ .

To find the next member of (4), for every  $i < \omega$ , let  $V_i \subseteq G_n$  be a complete set of representatives of the purification of  $(A_m + G_i)/(A_0 + G_i)$  in  $G/(A_0 + G_i)$ . The sets  $V_i$  are clearly countable, so that  $H_{m+1} = H_0 \cup \bigcup_{i < \omega} V_i$  is likewise countable. Therefore, there exists  $A_{m+1} \in \mathcal{B}_n$  containing both  $A_m$  and  $H_{m+1}$ , such that  $A_{m+1}/A_m$  is countable. Inductively, we construct a chain

$$(5) \quad A_0 < A_1 < \cdots < A_m < \cdots \quad (m < \omega)$$

of groups in  $\mathcal{B}_n$ , satisfying properties 1, 2 and 3 above, for every  $m < \omega$ .

Evidently, the union  $A$  of the links of (5) is a member of  $\mathcal{B}_n$ ,  $A/A_0$  is countable, and our construction guarantees that  $(A + G_i)/(A_0 + G_i)$  is pure in  $G/(A_0 + G_i)$ . Thus,  $A + G_i$  is pure in  $G$  and, consequently,  $A$  belongs to  $\mathcal{B}'_n$ .  $\square$

**Lemma 2.3.** *The collection  $\mathcal{B} = \{A < G : A \cap G_n \in \mathcal{B}'_n, \text{ for every } n < \omega\}$  is a  $G(\aleph_0)$ -family of pure subgroups of  $G$ .*

*Proof.* Again, only the countability condition merits attention; so, let  $A_0 \in \mathcal{B}$ , and let  $H \subseteq G$  be countable. For every  $k < \omega$ , let  $A_k^0 = A_0 \cap G_k$ . Moreover, let  $n < \omega$ , and assume that we have already constructed a finite ascending chain

$$(6) \quad A_0 < A_1 < \cdots < A_n$$

of subgroups of  $G$ , such that all factor groups  $A_m/A_0$  are countable, for every  $m \leq n$ . Furthermore, suppose that each link  $A_m$  in (6) may be expressed as the union of a countable ascending chain

$$(7) \quad 0 = A_0^m < A_1^m < \cdots < A_k^m < \cdots \quad (k < \omega)$$

of subgroups of  $G$ , such that:

- (a)  $A_k^m \in \mathcal{B}'_k$ , for every  $k < \omega$  and every  $m \leq n$ ,
- (b)  $A_k^m$  is countable over  $A_0 \cap G_k$ , for every  $k < \omega$  and every  $m \leq n$ , and
- (c)  $A_k^m < A_m \cap G_k < A_k^{m+1}$ , for every  $k < \omega$  and  $m + 1 \leq n$ .

For every  $k < \omega$ , the group  $(A_n \cap G_k)/(A_0 \cap G_k)$  is countable, so we may fix a countable set of representatives  $Y_k$  of  $A_n \cap G_k$  modulo  $A_0 \cap G_k$ . Moreover, there exists  $B_k \in \mathcal{B}'_k$  containing both  $A_0 \cap G_k$  and  $Y_k$ , such that  $B_k$  is countable over  $A_0 \cap G_k$ . Thus, any set of representatives  $H_k$  of  $B_k$  modulo  $A_0 \cap G_k$  is countable.

In order to construct the next link in (6), assume that the groups in the ascending chain  $0 = A_0^{n+1} < A_1^{n+1} < \cdots < A_k^{n+1}$  have been built as needed, for some  $k < \omega$ , and let  $Z_k \subseteq G_k$  be a set of representatives of  $A_k^{n+1}$  modulo  $A_0 \cap G_k$ . Then, there exists  $A_{k+1}^{n+1} \in \mathcal{B}'_{k+1}$  which contains  $A_0 \cap G_{k+1}$  and the countable set  $Z_k \cup H_{k+1} \cup (H \cap G_{k+1})$ , such that  $A_{k+1}^{n+1}$  is countable over  $A_0 \cap G_{k+1}$ .

Clearly, the group  $A = \bigcup_{n < \omega} A_n$  contains both  $A_0$  and  $H$ , and is countable over  $A_0$ . Moreover, our construction guarantees that  $A \cap G_k \in \mathcal{B}'_k$ , for every  $k < \omega$ . We conclude that  $A \in \mathcal{B}$ .  $\square$

Before we prove our next result, it is important to notice that  $A + G_n$  is a pure subgroup of  $G$ , for every  $A \in \mathcal{B}$  and every  $n < \omega$ . Indeed, that  $(A + G_n) \cap G_{n+1}$  is pure in  $G$  follows from the fact that  $A \cap G_{n+1} \in \mathcal{B}'_{n+1}$ . Next, assume that  $(A + G_n) \cap G_k$  is pure in  $G$ , for some  $k > n$ . It is easy to check that

$$(8) \quad \frac{(A + G_k) \cap G_{k+1}}{(A + G_n) \cap G_{k+1}} \cong \frac{G_k}{(A + G_n) \cap G_k},$$

whence it follows that  $(A + G_n) \cap G_{k+1}$  is pure in  $G$ . The claim is readily established after noticing that  $A + G_n = \bigcup_{k < \omega} (A + G_n) \cap G_k$ .

**Lemma 2.4.** *For every  $A \in \mathcal{B}$ , finite rank, pure subgroups of  $G/A$  are free.*

*Proof.* Let  $A \in \mathcal{B}$ , and let  $D$  be a pure subgroup of  $G$  containing  $A$ , such that  $D/A$  is of finite rank. If  $S = \{d_1, \dots, d_n\}$  is a complete set of representatives of a maximal independent system of  $D$  modulo  $A$ , then there exists  $k < \omega$  such that  $S \subseteq G_k$ . Then  $A + (D \cap G_k) = D \cap (A + G_k)$  is a pure subgroup of  $G$  containing  $S$ , which lies between  $A$  and  $D$ . Therefore,  $D = A + (D \cap G_k)$ . The fact that  $A \cap G_k \in \mathcal{B}'_k$  implies that  $A \cap G_k$  is a summand of  $G_k$ . Therefore, there exists a finite rank, free group  $B$ , such that  $D \cap G_k = (A \cap G_k) \oplus B$ . Notice that

$$(9) \quad D = A + (D \cap G_k) = A + ((A \cap G_k) \oplus B) = A \oplus B,$$

which implies that  $D/A$  is free.  $\square$

### 3. PROOF OF THE MAIN RESULT

*Proof of Theorem 1.1.* Let  $\alpha$  be any nonzero ordinal, and let

$$(10) \quad 0 = A_0 < A_1 < \dots < A_\gamma < A_{\gamma+1} \dots \quad (\gamma < \alpha)$$

be an ascending chain of subgroups in  $\mathcal{B}$ , such that all factor groups  $A_{\gamma+1}/A_\gamma$  are free. If  $\alpha$  is a limit ordinal, then we let  $A_\alpha = \bigcup_{\gamma < \alpha} A_\gamma$ . Otherwise, there exists an ordinal  $\beta$  such that  $\alpha = \beta + 1$ . In this case, if there exists  $x \in G \setminus A_\beta$ , we let  $A_{\beta+1} \in \mathcal{B}$  contain both  $x$  and  $A_\beta$ , such that  $A_{\beta+1}/A_\beta$  be countable. Lemma 2.4 implies now that finite rank, pure subgroups of  $A_{\beta+1}/A_\beta$  are free. Consequently,  $A_{\beta+1}/A_\beta$  is free by Pontryagin's criterion.

Using transfinite induction, we construct a continuous, well-ordered, ascending chain (3) of subgroups of  $G$  satisfying properties (a) and (b) of Lemma 2.1. We conclude that  $G$  is free.  $\square$

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